

HOMOLOGICAL STABILITY FOR MODULI SPACES OF HIGH DIMENSIONAL MANIFOLDS

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ABSTRACT. We prove a homological stability theorem for the moduli spaces of manifolds diffeomorphic to $\#^g S^n \times S^n$, provided $n > 2$. This generalises Harer's stability theorem for the homology of mapping class groups. Combined with previous work of the authors, it gives a calculation of the homology of these moduli spaces in a range of degrees.

1. INTRODUCTION AND STATEMENT OF RESULTS

A famous result of Harer ([Har85]) established *homological stability* for mapping class groups of oriented surfaces. For example, if $\Gamma_{g,1}$ denotes the group of isotopy classes of diffeomorphisms of an oriented connected surface of genus g with one boundary component, then the natural homomorphism $\Gamma_{g,1} \rightarrow \Gamma_{g+1,1}$ induces an isomorphism in group homology

$$H_k(\Gamma_{g,1}) \longrightarrow H_k(\Gamma_{g+1,1})$$

as long as $g \geq (3k + 2)/2$. (Harer proved this for $g \geq 3k - 1$, but the range was later improved by Ivanov ([Iva93]) and Boldsen ([Bol12]), see also [RW09].) This result can be interpreted in terms of moduli spaces of Riemann surfaces, and has lead to a wealth of research in topology and algebraic geometry. We prove an analogous homological stability result for moduli spaces of manifolds of higher (even) dimension. The precise result requires the following definition, where we assume $N \geq 2n$ and embed $S^{2n-1} \subset \mathbb{R}^{2n} \subset \mathbb{R}^N$ in the usual way.

Definition 1.1. Let $\mathcal{M}_g(\mathbb{R}^N) = \mathcal{M}_g^n(\mathbb{R}^N)$ denote the set of compact $2n$ -dimensional submanifolds $W \subset [0, \infty) \times \mathbb{R}^N$ such that $\partial W = \{0\} \times S^{2n-1}$ and $[0, \varepsilon) \times S^{2n-1} \subset W$ for some $\varepsilon > 0$, and such that W is diffeomorphic relative to its boundary to the manifold $W_{g,1} = \#^g(S^n \times S^n) - \text{int}(D^{2n})$. Topologise $\mathcal{M}_g(\mathbb{R}^N)$ as a quotient of the space of embeddings $W_{g,1} \hookrightarrow [0, \infty) \times \mathbb{R}^N$ (with fixed behaviour near the boundary).

For $N = \infty$ we write $\mathcal{M}_g = \text{colim}_{N \rightarrow \infty} \mathcal{M}_g(\mathbb{R}^N)$. Furthermore, we pick once and for all a (collared) embedding of the cobordism $W_{1,2} = S^n \times S^n - \text{int}(D^{2n} \amalg D^{2n})$ into $[0, 1] \times \mathbb{R}^N$. For $N \gg n$ all such embeddings are isotopic, and induce a well defined homotopy class of maps $\mathcal{M}_g \rightarrow \mathcal{M}_{g+1}$.

The space \mathcal{M}_g is a model for the classifying space $B\text{Diff}^\partial(W_{g,1})$ of the topological group of diffeomorphisms of $W_{g,1}$ fixing a neighbourhood of the boundary. For $n = 1$, this is an Eilenberg–MacLane space $K(\Gamma_{g,1}, 1)$, and hence Harer's result states that the stabilisation map $\mathcal{M}_g^1 \rightarrow \mathcal{M}_{g+1}^1$ induces an isomorphism in homology in a range. Our main result generalises this to higher n (although we exclude the case $n = 2$).

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Theorem 1.2. *For $n > 2$ the stabilisation map*

$$H_k(\mathcal{M}_g) \longrightarrow H_k(\mathcal{M}_{g+1})$$

is an isomorphism for $k \leq (g-4)/2$.

By the universal coefficient theorem, stability for homology implies stability for cohomology; in the surface case, Mumford ([Mum83]) conjectured an explicit formula for the stable rational cohomology, which in our notation asserts that a certain ring homomorphism

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \longrightarrow H^*(\mathcal{M}_g^1; \mathbb{Q})$$

is an isomorphism for $g \gg *$. Mumford’s conjecture was proved in a strengthened version by Madsen and Weiss ([MW07]).

Theorem 1.2 and our previous paper [GRW12] allow us to prove results analogous to Mumford’s conjecture and the Madsen–Weiss theorem for the moduli spaces \mathcal{M}_g^n with $n > 2$. The analogue of the Madsen–Weiss theorem for \mathcal{M}_g^n concerns the homology of the limiting space $\mathcal{M}_\infty^n = \text{colim}_{g \rightarrow \infty} \mathcal{M}_g^n$. There is a certain infinite loop space $\Omega^\infty MT\theta^n$ and a continuous map

$$\alpha : \mathcal{M}_\infty^n \longrightarrow \Omega^\infty MT\theta^n$$

given by a parametrised form of the Pontrjagin–Thom construction, and in [GRW12, Theorem 1.1] we proved that α induces an isomorphism between the homology of \mathcal{M}_∞^n and the homology of the basepoint component of $\Omega^\infty MT\theta^n$. It is easy to calculate the rational cohomology ring of a component of $\Omega^\infty MT\theta^n$, and hence of \mathcal{M}_g^n in a range of degrees by Theorem 1.2. The result is Corollary 1.3 below, which is a higher-dimensional analogue of Mumford’s conjecture.

As explained in [GRW12], we can associate to each $c \in H^{k+2n}(BSO(2n))$ a cohomology class $\kappa_c \in H^k(\Omega^\infty MT\theta^n)$. Pulling it back via α and all the stabilisation maps $\mathcal{M}_g^n \rightarrow \mathcal{M}_\infty^n$ defines classes $\kappa_c \in H^k(\mathcal{M}_g^n)$ for all g , sometimes called “generalised MMM classes”. In real cohomology these classes can equivalently be defined as follows. Suppose $\omega \in \Omega^{k+2n}(\text{Gr}_{2n}^+(\mathbb{R}^{N+1}))$ is a differential form and $\sigma : \Delta^k \rightarrow \mathcal{M}_g^n(\mathbb{R}^N)$ is a smooth map given by a $(k+2n)$ -dimensional manifold $W_\sigma \subset \Delta^k \times \mathbb{R}^N$ fibering over Δ^k . The fibrewise tangent spaces give a map $\tau_\sigma : W_\sigma \rightarrow \text{Gr}_{2n}^+(\mathbb{R}^{N+1})$ and hence a differential form $\tau_\sigma^* \omega \in \Omega^{k+2n}(W_\sigma)$, and we define

$$\kappa_c(\sigma) = \int_{W_\sigma} (\tau_\sigma^* \omega) \in \mathbb{R}.$$

We have defined a linear map

$$\Omega^{k+2n}(\text{Gr}_{2n}^+(\mathbb{R}^{N+1})) \longrightarrow C_{\text{sm}}^k(\mathcal{M}_g^n(\mathbb{R}^N); \mathbb{R})$$

which by Stokes’ theorem is a chain map (at least for $k \geq 0$). Hence, it induces a map of cohomology which in the limit $N \rightarrow \infty$ sends $c \in H^{k+2n}(BSO(2n); \mathbb{R})$ to $\kappa_c \in H^k(\mathcal{M}_g^n; \mathbb{R})$. The following result is our higher-dimensional analogue of Mumford’s conjecture.

Corollary 1.3. *Let $n > 2$ and let $\mathcal{B} \subset H^*(BSO(2n); \mathbb{Q})$ be the set of monomials in the classes $e, p_{n-1}, \dots, p_{\lceil \frac{n+1}{4} \rceil}$, of degree greater than $2n$. Then the induced map*

$$\mathbb{Q}[\kappa_c \mid c \in \mathcal{B}] \longrightarrow H^*(\mathcal{M}_g; \mathbb{Q})$$

is an isomorphism in the range $ \leq (g-4)/2$.*

For example, if $n = 3$, the set \mathcal{B} consists of monomials in e, p_1 and p_2 , and therefore $H^*(\mathcal{M}_g^3; \mathbb{Q})$ agrees for $* \leq (g-4)/2$ with a polynomial ring in variables of degrees 2, 2, 4, 6, 6, 6, 8, 8, 10, 10, 10, 12, 12, \dots

Our methods are similar to those used to prove many homological stability results for homology of discrete groups, namely to use a suitable action of the group on a simplicial complex. For example, Harer used the action of the mapping class group on the *arc complex* to prove his homological stability result. In our case the relevant groups are not discrete, so we use a simplicial space instead—the full diffeomorphism group of $W_{g,1}$ plays the same role for our stability result as the mapping class group in Harer’s (similar to the situation in [RW09]).

Independently, Berglund and Madsen ([BM12]) have obtained a result similar to our Theorem 1.2, for rational cohomology in the range $k \leq \min(n-3, (g-6)/2)$.

2. TECHNIQUES

In this section we collect the technical results needed to establish high connectivity of the relevant simplicial spaces. The main results are Theorem 2.4 and Corollary 2.8.

2.1. Cohen–Macaulay complexes. Recall from [HW10, Definition 3.4] that a simplicial complex K is *weakly Cohen–Macaulay* of dimension n if it is $(n-1)$ -connected and the link of any p -simplex is $(n-p-2)$ -connected. In this case, we write $wCM(K) \geq n$.

Lemma 2.1. *If $wCM(X) \geq n$ and $\sigma < X$ is a p -simplex, then $wCM(\text{Lk}(\sigma)) \geq n-p-1$.*

Proof. By assumption, $\text{Lk}(\sigma)$ is $((n-p-1)-1)$ -connected. If $\tau < \text{Lk}(\sigma)$ is a q -simplex, then

$$\text{Lk}_{\text{Lk}(\sigma)}(\tau) = \text{Lk}_X(\sigma * \tau)$$

is $((n-p-1)-q-2)$ -connected, since $\sigma * \tau$ is a $(p+q+1)$ -simplex, and hence its link in X is $(n-(p+q+1)-2)$ -connected. \square

Definition 2.2. Let us say that a simplicial map $f : X \rightarrow Y$ of simplicial complexes is *simplexwise injective* if its restriction to each simplex of X is injective, i.e. the image of any p -simplex of X is a (non-degenerate) p -simplex of Y .

Lemma 2.3. *Let $f : X \rightarrow Y$ be a simplicial map of simplicial complexes. Then the following conditions are equivalent.*

- (i) f is simplexwise injective,
- (ii) $f(\text{Lk}(\sigma)) \subset \text{Lk}(f(\sigma))$ for all simplices $\sigma < X$,
- (iii) $f(\text{Lk}(v)) \subset \text{Lk}(f(v))$ for all vertices $v \in X$,
- (iv) The image of any 1-simplex in X is a (non-degenerate) 1-simplex in Y .

Proof.

- (i) \Rightarrow (ii) If $\sigma = \{v_0, \dots, v_p\}$ and $v \in \text{Lk}(\sigma)$, then $\{v, v_0, \dots, v_p\} < X$ is a simplex, and therefore $\{f(v), f(v_0), \dots, f(v_p)\} < Y$ is a simplex. Since f is simplexwise injective, we must have $f(v) \notin f(\sigma)$, so $f(v) \in \text{Lk}(f(\sigma))$.
- (ii) \Rightarrow (iii) Trivial.
- (iii) \Rightarrow (iv) Let $\sigma = \{v_0, v_1\} < X$ be a 1-simplex, and assume for contradiction that $f(v_0) = f(v_1)$. Then we have $v_1 \in \text{Lk}(v_0)$ but $f(v_1) = f(v_0) \notin \text{Lk}(f(v_0))$, contradicting $f(\text{Lk}(v_0)) \subset \text{Lk}(f(v_0))$.
- (iv) \Rightarrow (i) Let $\sigma = \{v_0, \dots, v_p\} < X$ be a p -simplex and assume for contradiction that $f|_\sigma$ is not injective. This means that $f(v_i) = f(v_j)$ for some $i \neq j$, but then the restriction of f to the 1-simplex $\{v_i, v_j\}$ is not injective. \square

The following theorem generalises the “colouring lemma” of Hatcher and Wahl ([HW10, Lemma 3.1]), which is the special case where X is a simplex. The proof given below is an adaptation of theirs.

Theorem 2.4. *Let X be a simplicial complex and $f : \partial I^n \rightarrow |X|$ be a map which is simplicial with respect to some PL triangulation of ∂I^n . Then, if $wCM(X) \geq n$, the triangulation extends to a PL triangulation of I^n , and f extends to a simplicial map $g : I^n \rightarrow |X|$ with the property that $g(\text{Lk}(v)) \subset \text{Lk}(g(v))$ for each interior vertex $v \in I^n - \partial I^n$. In particular, g is simplexwise injective if f is.*

Proof. Since $|X|$ is in particular $(n-1)$ -connected, we may extend f to a continuous map $I^n \rightarrow |X|$ which, by the simplicial approximation theorem, may be assumed simplicial with respect to some PL triangulation of I^n extending the given triangulation on ∂I^n . Thus there is a PL homeomorphism $I^n \approx |K|$ for which the extension $h : I^n \rightarrow |X|$ is simplicial. Let us say that a simplex $\sigma < K$ is *bad* if any vertex $v \in \sigma$ is contained in a 1-simplex $\{v, v'\} \subset \sigma$ with $h(v) = h(v')$. We will describe a procedure which replaces the simplicial map $h : I^n \rightarrow X$ by a “better” one, by changing both the map h and the simplicial complex K , arriving at the desired map g in finitely many steps.

If all bad simplices are contained in ∂I^n , we are done. If not, let $\sigma < K$ be a bad simplex not contained in ∂I^n , of maximal dimension p . Then $p > 0$, and we must have $h(\text{Lk}(\sigma)) \subset \text{Lk}(h(\sigma))$, since otherwise we could join a simplex in $\text{Lk}(\sigma)$ to σ and get a bad simplex of larger dimension. Now $|\sigma| \subset |K| \approx I^n$, so h restricts to a map

$$\partial I^{n-p} \approx \text{Lk}(\sigma) \longrightarrow \text{Lk}(h(\sigma)).$$

The image $h(\sigma)$ is a simplex of dimension $\leq p-1$, since otherwise $h|_\sigma$ would be injective (in fact it has dimension $\leq (p-1)/2$ by badness), so $\text{Lk}(h(\sigma))$ has

$$wCM(\text{Lk}(h(\sigma))) \geq n - (p-1) - 1 = n - p,$$

and in particular $\text{Lk}(h(\sigma))$ is $(n-p-1)$ -connected, so $h|_{\text{Lk}(\sigma)}$ extends to a PL map

$$I^{n-p} \approx C(\text{Lk}(\sigma)) \xrightarrow{\tilde{h}} \text{Lk}(h(\sigma)).$$

By induction on n , we may assume that \tilde{h} is simplicial with respect to a PL triangulation of $C(\text{Lk}(\sigma))$ which extends the triangulation of $\text{Lk}(\sigma)$, and such that all bad simplices of \tilde{h} are in $\partial I^{n-p} = \text{Lk}(\sigma)$. We may extend this by joining with $h|_{\partial \sigma}$ to get a map

$$\sigma * \text{Lk}(\sigma) \approx (\partial \sigma) * (C\text{Lk}(\sigma)) \xrightarrow{\tilde{h}} X$$

which we may finally extend to I^n by setting it equal to h outside $\sigma * \text{Lk}(\sigma) \subset |K|$. The new map \tilde{h} has fewer bad simplices (not contained in ∂I^n) of dimension p . \square

2.2. Serre microfibrations. Let us recall from [Wei05] that a map $p : E \rightarrow B$ is called a *Serre microfibration* if for any k and any lifting diagram

$$\begin{array}{ccc} \{0\} \times D^k & \xrightarrow{f} & E \\ \downarrow & & \downarrow p \\ [0, 1] \times D^k & \xrightarrow{h} & B \end{array}$$

there exists an $\varepsilon > 0$ and a map $H : [0, \varepsilon] \times D^k \rightarrow E$ with $H(0, x) = f(x)$ and $p \circ H(t, x) = h(t, x)$ for all $x \in D^k$ and $t \in [0, \varepsilon]$. This condition implies that if (X, A) is a finite CW pair then any map $X \rightarrow B$ may be lifted in a neighbourhood of A , extending any prescribed lift over A . It also implies the following useful

observation: suppose (Y, X) is a finite CW pair and we are given a lifting problem

$$(2.1) \quad \begin{array}{ccc} X & \xrightarrow{f} & E \\ \downarrow & & \downarrow p \\ Y & \xrightarrow{F} & B. \end{array}$$

If there exists a map $G : Y \rightarrow E$ lifting F and so that $G|_X$ is fibrewise homotopic to f , then there is also a lift H of F so that $H|_X = f$. To see this, choose a fibrewise homotopy $\varphi : [0, 1] \times X \rightarrow E$ from $G|_X$ to f , let $J = ([0, 1] \times X) \cup (\{0\} \times Y) \subset [0, 1] \times Y$ and write $\varphi \cup G : J \rightarrow E$ for the map induced by φ and G . The following diagram is then commutative

$$\begin{array}{ccc} J & \xrightarrow{\varphi \cup G} & E \\ \downarrow & & \downarrow p \\ [0, 1] \times Y & \xrightarrow{\pi_Y} Y & \xrightarrow{F} B, \end{array}$$

and by the microfibration property there is a lift $g : U \rightarrow E$ defined on an open neighbourhood U of J . Let $\phi : Y \rightarrow [0, 1]$ be a continuous function with graph inside U and so that $X \subset \phi^{-1}(1)$. Then we set $H(y) = g(\phi(y), y)$; this is a lift of F as g is a lift of $F \circ \pi_Y$, and if $y \in X$ then $\phi(y) = 1$ and so $H(y) = g(1, y) = f(y)$, as required.

Examples of Serre microfibrations include submersions of manifolds, and when E is an open subspace of the total space of a Serre fibration (more generally, an open subset of another Serre microfibration). Weiss proved in [Wei05, Lemma 2.2] that if $f : E \rightarrow B$ is a Serre microfibration with weakly contractible fibres (i.e. $f^{-1}(b)$ is weakly contractible for all $b \in B$), then f is in fact a Serre fibration and hence a weak equivalence. We shall need the following generalisation, whose proof is essentially the same as Weiss'.

Proposition 2.5. *Let $p : E \rightarrow B$ be a Serre microfibration such that $p^{-1}(b)$ is n -connected for all $b \in B$. Then the homotopy fibres of p are also n -connected, i.e. p is $(n+1)$ -connected.*

Proof. Let us first prove that $p^I : E^I \rightarrow B^I$ is a Serre microfibration with $(n-1)$ -connected fibres, where $X^I = \text{Map}([0, 1], X)$ is the space of (unbased) paths in X , equipped with the compact-open topology. Using the mapping space adjunction, it is obvious that p^I is a Serre microfibration, and showing the connectivity of its fibres amounts to proving that any diagram of the form

$$\begin{array}{ccc} [0, 1] \times \partial D^k & \xrightarrow{\quad} & E \\ \downarrow & & \downarrow p \\ [0, 1] \times D^k & \xrightarrow{\text{proj}} [0, 1] & \longrightarrow B \end{array}$$

with $k \leq n$ admits a diagonal $h : [0, 1] \times D^k \rightarrow E$. Since fibres of p are $(k-1)$ -connected (in fact k -connected), such a diagonal can be found on each $\{a\} \times D^k$, and by the microfibration property these lifts extend to a neighbourhood. By the Lebesgue number lemma we may therefore find an integer $N \gg 0$ and lifts $h_i : [(i-1)/N, i/N] \times D^k \rightarrow E$ for $i = 1, \dots, N$. The two restrictions $h_i, h_{i+1} : \{i/N\} \times D^k \rightarrow E$ agree on $\{i/N\} \times \partial D^k$ and map into the same fibre of p . Since these fibres are k -connected, the restrictions of h_i and h_{i+1} are homotopic relative to $\{i/N\} \times \partial D^k$ as maps into the fibre, and we may use diagram (2.1) with $Y = [i/N, (i+1)/N] \times D^k$ and $X = (\{i/N\} \times D^k) \cup ([i/N, (i+1)/N] \times \partial D^k)$ to inductively replace h_{i+1} with a

homotopy which can be concatenated with h_i . The concatenation of the h_i 's then gives the required diagonal.

Let us now prove that for all $k \leq n$, any lifting diagram

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{f} & E \\ \downarrow & \nearrow H & \downarrow p \\ [0, 1] \times I^k & \xrightarrow{h} & B \end{array}$$

admits a diagonal map H making the diagram commutative. To see this, we first use that fibres of the map $p^{I^{k+1}} : E^{I^{k+1}} \rightarrow B^{I^{k+1}}$ are non-empty (in fact $(n-k-1)$ -connected) to find a diagonal G making the lower triangle commute. The restriction of G to $\{0\} \times I^k$ need not agree with f , but they lie in the same fiber of $p^{I^k} : E^{I^k} \rightarrow B^{I^k}$. Since this map has path connected fibres, these are fibrewise homotopic, and hence we may apply (2.1) to replace G with a lift H making both triangles commute.

This homotopy lifting property implies that the inclusion of $p^{-1}(b)$ into the homotopy fibre of p over b is n -connected, and hence that the homotopy fibre is n -connected. \square

2.3. Semisimplicial sets and spaces. Let Δ_{inj}^* be the category whose objects are the ordered sets $[p] = (0 < \dots < p)$ with $p \geq -1$, and whose morphisms are the injective, order preserving functions. An augmented semisimplicial set is a contravariant functor X from Δ_{inj}^* to the category of sets. As usual, such a functor is specified by the sets $X_p = X([p])$ and face maps $d_i : X_p \rightarrow X_{p-1}$ for $i = 0, \dots, p$. A (non-augmented) semisimplicial set is a functor defined on the full subcategory Δ_{inj} on the objects with $p \geq 0$. Semisimplicial spaces are defined similarly. We shall use the following well known result.

Proposition 2.6. *Let $f_\bullet : X_\bullet \rightarrow Y_\bullet$ be a map of semisimplicial spaces such that $f_p : X_p \rightarrow Y_p$ is $(n-p)$ -connected for all p . Then $|f_\bullet| : |X_\bullet| \rightarrow |Y_\bullet|$ is n -connected. \square*

Let us briefly discuss the relationship between simplicial complexes and semisimplicial sets. To any simplicial complex K there is an associated semisimplicial set K_\bullet , whose p -simplices are the injective simplicial maps $\Delta^p \rightarrow K$, i.e. ordered $(p+1)$ -tuples of vertices in K spanning a p -simplex. There is a natural surjection $|K_\bullet| \rightarrow |K|$, and any choice of total order on the set of vertices of K induces a splitting $|K| \rightarrow |K_\bullet|$. In particular, $|K|$ is at least as connected as $|K_\bullet|$.

Proposition 2.7. *Let Y_\bullet be a semisimplicial set, and Z be a Hausdorff space. Let $X_\bullet \subset Y_\bullet \times Z$ be a sub-semisimplicial space which in each degree is an open subset. Then $\pi : |X_\bullet| \rightarrow Z$ is a Serre microfibration.*

Proof. For $\sigma \in Y_n$, let us write $Z_\sigma \subset Z$ for the open subset defined by $(\{\sigma\} \times Z) \cap X_n = \{\sigma\} \times Z_\sigma$. Points in $|X_\bullet|$ are described by data

$$(\sigma \in Y_n; z \in Z_\sigma; (t_0, \dots, t_n) \in \Delta^n)$$

up to the evident relation when some t_i is zero, but we emphasise that the continuous, injective map $\iota = p \times \pi : |X_\bullet| \hookrightarrow |Y_\bullet| \times Z$ will not typically be a homeomorphism onto its image.

Suppose we have a lifting problem

$$\begin{array}{ccc} \{0\} \times D^k & \xrightarrow{f} & |X_\bullet| \\ \downarrow & & \downarrow \pi \\ [0, 1] \times D^k & \xrightarrow{F} & Z. \end{array}$$

The composition $D^k \xrightarrow{f} |X_\bullet| \xrightarrow{p} |Y_\bullet|$ is continuous, so the image of D^k is compact and hence contained in a finite subcomplex, and it intersects finitely many open simplices $\{\sigma_i\} \times \text{int}(\Delta^{n_i}) \subset |Y_\bullet|$. The sets $C_{\sigma_i} = (p \circ f)^{-1}(\{\sigma_i\} \times \text{int}(\Delta^{n_i}))$ then cover D^k , and their closures \overline{C}_{σ_i} give a finite cover of D^k by closed sets. Let us write $f|_{C_{\sigma_i}}(x) = (\sigma_i; z(x); t(x))$, with $z(x) \in Z_{\sigma_i} \subset Z$ and $t(x) = (t_0(x), \dots, t_{n_i}(x)) \in \text{int}(\Delta^{n_i})$.

Certainly $\pi \circ f$ sends the set C_{σ_i} into the open set Z_{σ_i} , but we claim that \overline{C}_{σ_i} is also mapped into Z_{σ_i} . To see this, we consider a sequence $(x^j) \in C_{\sigma_i}$, $j \in \mathbb{N}$ converging to a point $x \in \overline{C}_{\sigma_i} \subset D^k$ and verify that $z = \pi \circ f(x) \in Z_{\sigma_i}$. As f is continuous, the sequence $f(x^j) = (\sigma_i; z(x^j); t(x^j)) \in |X_\bullet|$ converges to $f(x)$, and passing to a subsequence, we may assume that the $t(x^j)$ converge to a point $t \in \Delta^{n_i}$. The subset

$$A = \{f(x^j) \mid j \in \mathbb{N}\} \subset |X_\bullet|$$

is contained in $\pi^{-1}(Z_{\sigma_i})$ and has $f(x)$ as a limit point in $|X_\bullet|$, so if $z = \pi(f(x)) \notin Z_{\sigma_i}$, the set A is not closed in $|X_\bullet|$. For a contradiction, we will show that A is closed, by proving that its inverse image in $\coprod_\tau \{\tau\} \times Z_\tau \times \Delta^{|\tau|}$ is closed, where the coproduct is over all simplices $\tau \in \coprod_n Y_n$. The inverse image in $\{\sigma_i\} \times Z_{\sigma_i} \times \Delta^{n_i}$ is

$$B = \{(\sigma_i; z(x^j); t(x^j)) \mid j \in \mathbb{N}\},$$

which is closed (since Z is Hausdorff, taking the closure in $\{\sigma_i\} \times Z \times \Delta^{n_i}$ adjoins only the point $(\sigma_i; z; t)$, which by assumption is outside $\{\sigma_i\} \times Z_{\sigma_i} \times \Delta^{n_i}$). If $\sigma_i = \theta^*(\tau)$ for a morphism $\theta \in \Delta_{\text{inj}}$, we have $Z_\tau \subset Z_{\sigma_i}$ and hence $B \cap (\{\sigma_i\} \times Z_\tau \times \Delta^{|\sigma_i|})$ is closed in $\{\sigma_i\} \times Z_\tau \times \Delta^{|\sigma_i|}$ so applying $\theta_* : \Delta^{|\sigma_i|} \rightarrow \Delta^{|\tau|}$ gives a closed subset $B_\theta \subset \{\tau\} \times Z_\tau \times \Delta^{|\tau|}$. The inverse image of A in $\{\tau\} \times Z_\tau \times \Delta^{|\tau|}$ is the union of the B_θ over the finitely many θ with $\theta^*(\tau) = \sigma_i$, and is hence closed.

We have a continuous map $F_i = F|_{[0,1] \times \overline{C}_{\sigma_i}} : [0,1] \times \overline{C}_{\sigma_i} \rightarrow Z$ and $F_i^{-1}(Z_{\sigma_i})$ is an open neighbourhood of the compact set $\{0\} \times \overline{C}_{\sigma_i}$, so there is an $\varepsilon_i > 0$ such that $F_i([0, \varepsilon_i] \times \overline{C}_{\sigma_i}) \subset Z_{\sigma_i}$. We set $\varepsilon = \min_i(\varepsilon_i)$ and define the lift

$$\tilde{F}_i(s, x) = (\sigma_i; F_i(s, x); t(x)) : [0, \varepsilon] \times \overline{C}_{\sigma_i} \longrightarrow \{\sigma_i\} \times Z_{\sigma_i} \times \Delta^{n_i} \longrightarrow |X_\bullet|,$$

which is clearly continuous. The functions \tilde{F}_i and \tilde{F}_j agree where they are both defined, and so these glue to give a continuous lift \tilde{F} as required. \square

Corollary 2.8. *Let Z , Y_\bullet , and X_\bullet be as in Proposition 2.7. For $z \in Z$, let $X_\bullet(z) \subset Y_\bullet$ be the sub-semisimplicial set defined by $X_\bullet \cap (Y_\bullet \times \{z\}) = X_\bullet(z) \times \{z\}$ and suppose that $|X_\bullet(z)|$ is n -connected for all $z \in Z$. Then the map $\pi : |X_\bullet| \rightarrow Z$ is $(n+1)$ -connected.*

Proof. This follows by combining Propositions 2.5 and 2.7, once we prove that $|X_\bullet(z)|$ is homeomorphic to $\pi^{-1}(z)$ (in the subspace topology from $|X_\bullet|$). Since $X_\bullet(z) \subset Y_\bullet$, the composition $|X_\bullet(z)| \rightarrow |X_\bullet| \rightarrow |Y_\bullet|$ is a homeomorphism onto its image. It follows that $|X_\bullet(z)| \rightarrow |X_\bullet|$ is a homeomorphism onto its image, which is easily seen to be $\pi^{-1}(z)$. \square

3. ALGEBRA

We fix $\varepsilon = \pm 1$. Let $\Lambda \subset \mathbb{Z}$ be a subgroup satisfying

$$\{a - \varepsilon a \mid a \in \mathbb{Z}\} \subset \Lambda \subset \{a \in \mathbb{Z} \mid a + \varepsilon a = 0\}.$$

Following Bak ([Bak69, Bak81]), we call such a pair (ε, Λ) a *form parameter*. An (ε, Λ) -*Quadratic module* (M, λ, α) is the data of a \mathbb{Z} -module M , a ε -symmetric bilinear form

$$\lambda : M \otimes M \longrightarrow \mathbb{Z}$$

and a Λ -quadratic form

$$\alpha : M \longrightarrow \mathbb{Z}/\Lambda$$

whose associated bilinear form is λ reduced modulo Λ . By this we mean a function α such that

- (i) $\alpha(a \cdot x) = a^2 \cdot \alpha(x)$ for $a \in \mathbb{Z}$,
- (ii) $\alpha(x + y) = \alpha(x) + \alpha(y) + \lambda(x, y)$.

We say the (ε, Λ) -Quadratic module is *non-degenerate* if the map

$$\begin{aligned} M &\longrightarrow M^* \\ x &\longmapsto \lambda(-, x) \end{aligned}$$

is an isomorphism. A *morphism* of (ε, Λ) -Quadratic modules is a homomorphism $f : M \rightarrow N$ of modules which is an isometry for λ , and such that $\alpha_M = \alpha_N \circ f$. If M is non-degenerate, any such morphism is injective, as

$$M \xrightarrow{f} N \longrightarrow N^* \xrightarrow{f^*} M^*$$

is an isomorphism. The *hyperbolic module* H is the (ε, Λ) -Quadratic module given by the data

$$\left(\mathbb{Z}^2 \text{ with basis } e, f; \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}; \alpha(e) = \alpha(f) = 0 \right).$$

Definition 3.1. For an (ε, Λ) -Quadratic module (M, λ, α) , let $K^a(M)$ be the simplicial complex whose vertices are morphisms $e : H \rightarrow M$ of quadratic modules. The set $\{e_0, \dots, e_p\}$ is a p -simplex if the submodules $e_i(H) \subset M$ are orthogonal with respect to λ (and no condition on the quadratic forms).

If $\sigma = \{v_0, \dots, v_p\} \subset K^a(M)$, then the link $\text{Lk}(\sigma)$ is isomorphic to $K^a(M \cap \text{span}(v_0, \dots, v_p)^\perp)$.

This complex is almost the same as one considered by Charney, which she proves to be highly connected.

Theorem 3.2. *Let $M = H^{\oplus g}$. Then $|K^a(M)|$ is $\lfloor (g-5)/2 \rfloor$ -connected.*

Proof. The simplicial complex $K^a(M)$ has an associated semisimplicial set $K^\bullet_a(M)$, and in [Cha87, Corollary 3.3] it is proved that (the barycentric subdivision of) $|K^\bullet_a(M)|$ is $\lfloor (g-5)/2 \rfloor$ -connected. As mentioned in Section 2.3, this implies that $|K^a(M)|$ is also $\lfloor (g-5)/2 \rfloor$ -connected. \square

Corollary 3.3 (Transitivity). *If $e_0, e_1 : H \rightarrow H^{\oplus g}$ are morphisms of quadratic modules and $g \geq 5$, there is an isomorphism of quadratic modules $f : H^{\oplus g} \rightarrow H^{\oplus g}$ such that $e_1 = f \circ e_0$.*

Proof. Suppose first that e_0 and e_1 are orthogonal. Then

$$H^{\oplus g} \cong e_0(H) \oplus e_1(H) \oplus M$$

and there is an evident automorphism of quadratic modules which swaps the $e_i(H)$. Now, the relation between morphisms $e : H \rightarrow H^{\oplus g}$ of differing by an automorphism is an equivalence relation, and we have just shown that adjacent vertices in $K^a(H^{\oplus g})$ are equivalent. When $g \geq 5$ this simplicial complex is connected, and so all vertices are equivalent. \square

Corollary 3.4 (Cancellation). *Suppose that M is a quadratic module and there is an isomorphism $M \oplus H \cong H^{\oplus g+1}$ for $g \geq 4$. Then $M \cong H^{\oplus g}$.*

Proof. An isomorphism $\varphi : M \oplus H \rightarrow H^{\oplus g+1}$ gives a morphism $\varphi|_H : H \rightarrow H^{\oplus g+1}$ of quadratic modules, and we also have the standard inclusion $e_{g+1} : H \rightarrow H^{\oplus g+1}$. When $g+1 \geq 5$, the previous corollary shows that these differ by an automorphism of $H^{\oplus g+1}$, and in particular their orthogonal complements are isomorphic. \square

Corollary 3.5. *Let $M = H^{\oplus g}$. Then $wCM(K^a(M)) \geq \lfloor (g-3)/2 \rfloor$.*

Proof. We prove the statement assuming $g \geq 4$ (it is obvious for $g \leq 4$). Let $\sigma = \{v_0, \dots, v_p\} < K^a(H^{\oplus g})$ be a p -simplex, with link $K^a(\text{span}(e_0, \dots, e_p)^\perp)$. As

$$H^{\oplus g} \cong e_0(H) \oplus \dots \oplus e_p(H) \oplus \text{span}(e_0, \dots, e_p)^\perp$$

by Corollary 3.4, $K^a(\text{span}(e_0, \dots, e_p)^\perp)$ is isomorphic to the complex $K^a(H^{\oplus g-p-1})$, as long as $g-p-1 \geq 4$. In this case the link is then $\lfloor (g-p-6)/2 \rfloor$ -connected, but

$$\lfloor (g-p-6)/2 \rfloor \geq \lfloor (g-3)/2 \rfloor - p - 2,$$

which proves the claim. In the case $g-p-1 < 4$ we have $\lfloor (g-3)/2 \rfloor - p - 2 \leq \lfloor (1-g)/2 \rfloor \leq -2$, so there is no condition on the link in this case. \square

4. TOPOLOGY

We fix a dimension $d = 2n \geq 6$ throughout and write

$$W_g = \#^g S^n \times S^n$$

for the g -fold connected sum of $S^n \times S^n$ with itself. This is a $2n$ -dimensional smooth closed manifold, and we write $W_{g,k}$ for the manifold with boundary obtained by removing the interiors of k disjoint discs from W_g . (Up to diffeomorphism, the manifold $W_{g,k}$ does not depend on choices of where to perform connected sum and which discs to cut out. We make these choices once and for all, and the notation $W_{g,k}$ will denote an actual abstract manifold, with boundary components parametrised by S^{2n-1} , rather than a diffeomorphism class.) It will be convenient to have available the following small modification of the manifold $W_{1,1}$. Let H denote the manifold obtained from $W_{1,1} = S^n \times S^n - \text{int}(D^{2n})$ by gluing $[0, 1] \times D^{2n-1}$ onto $\partial W_{1,1}$ along an oriented embedding

$$\{1\} \times D^{2n-1} \longrightarrow \partial W_{1,1},$$

which we also choose once and for all. This gluing of course doesn't change the diffeomorphism type (after smoothing corners), so H is diffeomorphic to $W_{1,1}$, but contains a standard embedded $[0, 1] \times D^{2n-1} \subset H$. When we discuss embeddings of H into a manifold with boundary W , we shall always insist that $\{0\} \times D^{2n-1}$ is sent into ∂W , and that the rest of H is sent into the interior of W .

We shall also need a *core* $C \subset H$, defined as follows. Let $x_0 \in S^n$ be a basepoint. Let $S^n \vee S^n = (S^n \times \{x_0\}) \cup (\{x_0\} \times S^n) \subset S^n \times S^n$, which we may suppose is contained in $\text{int}(W_{1,1})$. Choose an embedded path γ in $\text{int}(H)$ from $(x_0, -x_0)$ to $(0, 0) \in [0, 1] \times D^{2n-1}$ whose interior does not intersect $S^n \vee S^n$, and whose image agrees with $[0, 1] \times \{0\}$ inside $[0, 1] \times D^{2n-1}$, and let

$$C = (S^n \vee S^n) \cup \gamma([0, 1]) \cup (\{0\} \times D^{2n-1}) \subset H.$$

We may choose an isotopy of embeddings $\rho_t : H \rightarrow H$, defined for $t \in [0, \infty)$, which starts at the identity, eventually has image inside any given neighbourhood of C , and which for each t is the identity on some neighbourhood of C .

Definition 4.1. Let W be a compact manifold, equipped with a fixed embedding $c : [0, 1] \times \mathbb{R}^{2n-1} \rightarrow W$ such that $c^{-1}(\partial W) = \{0\} \times \mathbb{R}^{2n-1}$.

- (i) Let $K_0(W) = K_0(W, c)$ be the space of pairs (t, ϕ) , where $t \in \mathbb{R}$ and $\phi : H \rightarrow W$ is an embedding whose restriction to $[0, 1] \times D^{2n-1} \subset H$ satisfies that there exists an $\varepsilon > 0$ such that

$$\phi(s, p) = c(s, p + te_1)$$

for all $s < \varepsilon$ and all $p \in D^{2n-1}$. Here, $e_1 \in \mathbb{R}^{2n-1}$ denotes the first basis vector.

- (ii) Let $K_p(W) \subset (K_0(W))^{p+1}$ consist of those tuples $((t_0, \phi_0), \dots, (t_p, \phi_p))$ satisfying that $t_0 < \dots < t_p$ and that the embeddings ϕ_i have disjoint cores, i.e. the sets $\phi_i(C)$ are disjoint.
- (iii) Topologise $K_p(W)$ using the C^∞ -topology on the space of embeddings and let $K_p^\delta(W)$ be the same set considered as a discrete topological space.
- (iv) The assignments $[p] \mapsto K_p(W)$ and $[p] \mapsto K_p^\delta(W)$ define semisimplicial spaces, where the face map d_i forgets (t_i, ϕ_i) .
- (v) Let $K^\delta(W)$ be the simplicial complex with vertices $K_0^\delta(W)$, and where the (unordered) set $\{(t_0, \phi_0), \dots, (t_p, \phi_p)\}$ is a p -simplex if, when written with $t_0 < \dots < t_p$, it satisfies $((t_0, \phi_0), \dots, (t_p, \phi_p)) \in K_p^\delta(W)$.

We shall often denote a vertex (t, ϕ) simply by ϕ , since t is determined by ϕ . Since a p -simplex of $K^\delta(W)$ is determined by its (unordered) set of vertices, there is a natural homeomorphism $|K^\delta(W)| = |K^\delta(W)|$.

The fibration $S^n \rightarrow BO(n) \rightarrow BO(n+1)$ gives an exact sequence

$$\dots \rightarrow \pi_{n+1}(BO(n+1)) \xrightarrow{\partial} \pi_n(S^n) = \mathbb{Z} \xrightarrow{\tau} \pi_n(BO(n)) \xrightarrow{s} \pi_n(BO(n+1)) \rightarrow 0$$

and we define $\Lambda_n := \text{Im}(\partial) \subset \mathbb{Z}$. This can of course be explicitly computed: it is 0 if n is even (the Euler class detects the injectivity of τ), \mathbb{Z} if $n \in \{1, 3, 7\}$, and $2\mathbb{Z}$ otherwise, by the Hopf invariant 1 theorem.

The data $((-1)^n, \Lambda_n)$ is a *form parameter*, in the sense of Section 3. Following Wall ([Wal62]), we will now construct from a stably parallelisable, $(n-1)$ -connected $2n$ -manifold W , a quadratic module having this form parameter. The first non-zero homotopy group of such a manifold is

$$\pi_n(W) \cong H_n(W; \mathbb{Z}).$$

Using the intersection form on the middle homology of W , we obtain a bilinear form

$$\lambda : \pi_n(W) \otimes \pi_n(W) \rightarrow \mathbb{Z}$$

which is $(-1)^n$ -symmetric, and non-degenerate by Poincaré duality. By a theorem of Haefliger ([Hae61]), an element $x \in \pi_n(W)$ may be represented by an embedded sphere as long as $n \geq 3$, and this representation is unique up to isotopy as long as $n \geq 4$. Such an embedding has a normal bundle which is stably trivial, so is represented by an element

$$\alpha(x) \in \text{Ker} \left(\pi_n(BO(n)) \xrightarrow{s} \pi_n(BO) \right) = \mathbb{Z}/\Lambda_n.$$

This gives a well-defined function $\alpha : \pi_n(W) \rightarrow \mathbb{Z}/\Lambda_n$ (it is well-defined even when $n = 3$, as $\mathbb{Z}/\Lambda_3 = \{0\}$).

Lemma 4.2. *The data $(\pi_n(W), \lambda, \alpha)$ is a $((-1)^n, \Lambda_n)$ -Quadratic module.*

Proof. See [Wal62, Lemma 2]. □

If we start with the manifold $W_{g,1} = \#^g S^n \times S^n \setminus \text{int}(D^{2n})$ then the associated quadratic module is isomorphic to $H^{\oplus g}$. Furthermore, given an embedding $e : H \rightarrow W_{g,1}$ we have an induced morphism $E : H \rightarrow \pi_n(W_{g,1})$ of quadratic modules, and disjoint embeddings give orthogonal morphisms. This defines a map of simplicial complexes

$$(4.1) \quad K^\delta(W_{g,1}) \rightarrow K^a(\pi_n(W_{g,1}), \lambda, \alpha)$$

which we will use to compute the connectivity of $|K^\delta(W_{g,1})| = |K^\delta(W_{g,1})|$. For brevity we shall just write $K^\delta \rightarrow K^a$ for this map in the proof of the following result.

Lemma 4.3. *The space $|K^\delta(W_{g,1})|$ is $\lfloor (g-5)/2 \rfloor$ -connected.*

Proof. Let $k \leq (g-5)/2$ and consider a map $f : S^k \rightarrow |K^\delta|$, which we may assume is simplicial with respect to some PL triangulation of $S^k = \partial I^{k+1}$. By Theorem 3.2, the composition $\partial I^{k+1} \rightarrow |K^\delta| \rightarrow |K^a|$ is null-homotopic and so extends to a map $g : I^{k+1} \rightarrow |K^a|$, which we may suppose is simplicial with respect to a PL triangulation of I^{k+1} extending the triangulation of its boundary.

By Corollary 3.5, we have $wCM(K^a) \geq \lfloor (g-3)/2 \rfloor$. By Theorem 2.4, as $k+1 \leq \lfloor (g-3)/2 \rfloor$ we can arrange that g is simplexwise injective on the interior of I^{k+1} . Then we pick a total order on the interior vertices, and inductively pick lifts of each vertex to K_0^δ . In each step, a vertex is given by a morphism of quadratic modules $J : H \rightarrow \pi_n(W_{g,1})$. The element $J(e)$ is represented by a map $x : S^n \rightarrow W_{g,1}$ which by Haefliger's theorem is representable by an embedding, and as $\alpha(x) = \alpha(e) = 0$ this embedding has trivial normal bundle. Thus $J(e)$ can be represented by an embedding $j(e) : S^n \times D^n \rightarrow W_{g,1}$. Similarly, $J(f)$ can be represented by an embedding $j(f) : S^n \times D^n \rightarrow W_{g,1}$.

As $\lambda(J(e), J(f)) = 1$, these two embeddings have algebraic intersection number 1. As $W_{g,1}$ is simply-connected and of dimension at least 6, we may use the Whitney trick to isotope these embeddings so that their cores $S^n \times \{0\}$ intersect transversely in precisely one point, and so obtain an embedding of the plumbing of $S^n \times D^n$ and $D^n \times S^n$, which is diffeomorphic to $W_{1,1} \subset H$. To extend to the remaining $[0, 1] \times D^{2n-1} \subset H$, we first pick an embedding $\{0\} \times D^{2n-1} \rightarrow \partial W_{g,1}$ disjoint from previous embeddings and satisfying condition (i) of Definition 4.1; then extend to an embedding of $[0, 1] \times \{0\}$ which we thicken to an embedding of $[0, 1] \times D^{2n-1}$. Finally, as J is orthogonal to any adjacent vertices which have already been lifted, we can use the Whitney trick again to isotope j so that its core is disjoint from the cores of all previously chosen vertices that are adjacent to it. After applying this procedure to all vertices, we obtain a lift of g to a null-homotopy of f , as required. \square

We now make deductions from the fact that $|K_\bullet^\delta(W_{g,1})|$ is path connected when $g \geq 5$, similar to those made about $K^a(H^{\oplus g})$ in Section 3.

Corollary 4.4 (Transitivity). *Let $e_0, e_1 : H \hookrightarrow W_{g,1}$ be embeddings, and $g \geq 5$. Then there is a diffeomorphism f of $W_{g,1}$ which is isotopic to the identity on the boundary and such that $e_1 = f \circ e_0$.*

Proof. Suppose first that e_0 and e_1 are disjoint. Let V denote the closure of a regular neighbourhood of $e_0(H) \cup e_1(H) \cup \partial W_{g,1}$, which is abstractly diffeomorphic to $W_{2,2}$ with two standard copies of H embedded in it, both connected to the first boundary. It is enough to find a diffeomorphism of $W_{2,2}$ which is the identity on the first boundary, is isotopic to the identity on the second boundary, and sends the first H to the second.

We give a concrete construction of such a diffeomorphism. Let $\Gamma \in SO(2n)$ be the diagonal matrix with entries $(-1, -1, 1, 1, \dots, 1)$. Let $d_0 : D^{2n} \rightarrow [0, 1] \times S^{2n-1}$ be a small disc around $(\frac{1}{2}, (1, 0, \dots, 0))$, so $d_1 = (\text{Id}_{[0,1]} \times \Gamma) \cdot d_0$ is a small disc around $(\frac{1}{2}, (-1, 0, \dots, 0))$. We form a manifold by connect summing two copies of $S^n \times S^n$ to $[0, 1] \times S^{2n-1}$ at these two discs,

$$M_{2,2} = (S^n \times S^n) \#_{d_0} ([0, 1] \times S^{2n-1}) \#_{d_1} (S^n \times S^n).$$

There is a diffeomorphism φ of $M_{2,2}$ which is given by $\text{Id}_{[0,1]} \times \Gamma$ on $[0, 1] \times S^{2n-1} \setminus (d_0(D^{2n}) \cup d_1(D^{2n}))$, and interchanges the two copies of $S^n \times S^n \setminus D^{2n}$. There is an embedded copy of H given by the first $S^n \times S^n \setminus D^{2n}$ along with a thickening of the arc $[\frac{1}{2}, 1] \times \{(1, 0, \dots, 0)\}$, and $\varphi(H)$ gives another disjoint embedded copy of H . We have found the required diffeomorphism, except that it is not the identity on the boundary $\{0\} \times S^{2n-1}$. To amend this, we replace $\text{Id}_{[0,1]} \times \Gamma$ in the above

construction with a function of the form $(t, x) \mapsto (t, \gamma(t)x)$, where $\gamma : [0, 1] \rightarrow SO(2n)$ is a path which is the identity on $[0, \varepsilon]$ and Γ on $[2\varepsilon, 1]$ for some small ε .

Now suppose that the e_i merely have disjoint cores, and recall the isotopy of self-embeddings $\rho_t : H \rightarrow H$ from before. For $T \gg 0$ the embeddings $e_i \circ \rho_T$ are disjoint, so by the Isotopy Extension Theorem we find diffeomorphisms $\varphi_i : W_{g,1} \rightarrow W_{g,1}$ (which are isotopic to the identity) such that the embeddings $\varphi_i \circ e_i$ are disjoint. By the above case we then find a diffeomorphism g of $W_{g,1}$ such that $(\varphi_1 \circ e_1) = g \circ (\varphi_0 \circ e_0)$, so $f = \varphi_1^{-1} \circ g \circ \varphi_0$ gives the diffeomorphism we require.

To prove the general case, when the e_i are not assumed to have disjoint cores, we use the connectedness of $|K_\bullet^\delta(W_{g,1})|$ when $g \geq 5$ and the argument of Corollary 3.3. \square

Corollary 4.5 (Cancellation). *Let M be a $2n$ -manifold with boundary parametrised by S^{2n-1} , and suppose there is a diffeomorphism*

$$\varphi : M \# S^n \times S^n \longrightarrow W_{g+1,1}$$

which is the identity on the boundary. Then if $g \geq 4$ there is a diffeomorphism of M with $W_{g,1}$ which is the identity on the boundary.

Proof. This is completely analogous to Corollary 3.4. We have the two embeddings $\varphi|_H : H \rightarrow W_{g+1,1}$ and $e_{g+1} : H \rightarrow W_{g+1,1}$, and by Corollary 4.4 there is a diffeomorphism f of $W_{g+1,1}$ isotopic to the identity on the boundary so that $e_{g+1} = f \circ \varphi|_H$. We obtain a diffeomorphism $f \circ \varphi : M \# S^n \times S^n \longrightarrow W_{g+1,1}$ isotopic to the identity on the boundary and which sends H to H identically. In particular it sends $W_{1,1} \subset H$ to itself identically, so after removing the interior of $W_{1,1}$ gives a diffeomorphism

$$M \setminus D^{2n} \longrightarrow W_{g,1} \setminus D^{2n}$$

which is isotopic to the identity on the old boundary, and is equal to the identity on the new boundary. We can then fill in the new boundary with a standard disc, and use isotopy extension to make the diffeomorphism be the identity on the remaining boundary. \square

In fact, Kreck ([Kre99, Theorem D]) proved the above cancellation result for $g \geq 1$ (and $g \geq 0$ when n is odd), but we shall only use the weaker result in Corollary 4.5.

Finally, we compare $|K_\bullet^\delta(W_{g,1})|$ and $|K_\bullet(W_{g,1})|$. The bisemisimplicial space in Definition 4.7 below will be used to leverage the known connectivity of $|K_\bullet^\delta(W_{g,1})|$ to prove the following theorem, which is the main result of this section.

Theorem 4.6. *The space $|K_\bullet(W_{g,1})|$ is $\lfloor (g-5)/2 \rfloor$ -connected.*

Definition 4.7. With W and c as in Definition 4.1, let $D_{p,q} = K_{p+q+1}(W)$, topologised as a subspace of $K_p(W) \times K_q^\delta(W)$. This is a bisemisimplicial space, equipped with augmentations

$$\begin{aligned} D_{p,q} &\xrightarrow{\varepsilon} K_p(W) \\ D_{p,q} &\xrightarrow{\delta} K_q^\delta(W). \end{aligned}$$

Lemma 4.8. *Let $\iota : K_\bullet^\delta(W) \rightarrow K_\bullet(W)$ denote the identity function. Then*

$$|\iota| \circ |\delta| \simeq |\varepsilon| : |D_{\bullet,\bullet}| \longrightarrow |K_\bullet(W)|.$$

Proof. For each p and q there is a homotopy

$$\begin{aligned} [0, 1] \times \Delta^p \times \Delta^q \times D_{p,q} &\longrightarrow \Delta^{p+q+1} \times K_{p+q+1}(W) \\ (r, s, t, x, y) &\longmapsto ((rs, (1-r)t), (x, \iota y)), \end{aligned}$$

where we write $(x, y) \in D_{p,q} \subset K_p(W) \times K_q^\delta(W)$ and $(x, \iota y) \in K_{p+q+1}(W) \subset K_p(W) \times K_q(W)$ and $(rs, (1-r)t) = (rs_0, \dots, rs_p, (1-r)t_0, \dots, (1-r)t_q) \in \Delta^{p+q+1}$. These homotopies glue to a homotopy $[0, 1] \times |D_{\bullet, \bullet}| \rightarrow |K_{\bullet}(W)|$ which starts at $|\iota| \circ |\delta|$ and ends at $|\varepsilon|$. \square

Proof of Theorem 4.6. We will apply Corollary 2.8 with $Z = K_p(W_{g,1})$, $Y_{\bullet} = K_{\bullet}^\delta(W_{g,1})$ and $X_{\bullet} = D_{p, \bullet}$. For $z = ((t_0, \phi_0), \dots, (t_p, \phi_p)) \in K_p(W_{g,1})$, we shall write $W_z \subset W$ for the complement of the $\phi_i(C)$. The realisation of the semisimplicial subset $X_{\bullet}(z) \subset Y_{\bullet} = K_{\bullet}^\delta(W_{g,1})$ is homeomorphic to the full subcomplex $F(z) \subset K^\delta(W_{g,1})$ on those (t, ϕ) such that $\phi(C) \subset W_z$ and $t > t_p$. The map of simplicial complexes (4.1) restricts to a map

$$F(z) \longrightarrow K^a(\pi_n(W_z), \lambda, \alpha).$$

By Corollary 3.4 the target is isomorphic to $K^a(H^{\oplus g-p-1})$ which is $\lfloor (g-p-6)/2 \rfloor$ -connected by Theorem 3.2, and the argument of Lemma 4.3 shows that $F(z)$ is too. (Corollary 3.4 only applies when $g-p-1 \geq 4$, but the statement is vacuously true for $g-p \leq 5$.) By Corollary 2.8, the map $|\varepsilon| : |D_{p, \bullet}| \rightarrow K_p(W_{g,1})$ is $\lfloor (g-p-4)/2 \rfloor$ -connected and since $\lfloor (g-p-4)/2 \rfloor \geq \lfloor (g-4)/2 \rfloor - p$, we deduce by Proposition 2.6 that the map $|D_{\bullet, \bullet}| \rightarrow |K_{\bullet}(W_{g,1})|$ is $\lfloor (g-4)/2 \rfloor$ -connected. But up to homotopy it factors through the $\lfloor (g-5)/2 \rfloor$ -connected space $|K_{\bullet}^\delta(W_{g,1})|$, and therefore $|K_{\bullet}(W_{g,1})|$ is $\lfloor (g-5)/2 \rfloor$ -connected too. \square

Finally, define the sub-semisimplicial space $\overline{K}_{\bullet}(W_{g,1}) \subset K_{\bullet}(W_{g,1})$ whose p -simplices are tuples of disjoint embeddings. (Recall that in $K_{\bullet}(W_{g,1})$ we only ask for the embeddings to have disjoint cores.)

Corollary 4.9. *The space $|\overline{K}_{\bullet}(W_{g,1})|$ is $\lfloor (g-5)/2 \rfloor$ -connected.*

Proof. Precomposing with the isotopy ρ_t , any tuple of embeddings with disjoint cores eventually become disjoint. It follows that the inclusion is a levelwise weak equivalence. \square

5. RESOLUTIONS OF MODULI SPACES

We now use the high connectivity of $|\overline{K}_{\bullet}(W_{g,1})|$ to prove Theorem 1.2.

5.1. A semisimplicial resolution. We shall define a semisimplicial resolution of the moduli space \mathcal{M}_g , meaning a semisimplicial space X_{\bullet} with an augmentation $X_{\bullet} \rightarrow \mathcal{M}_g$ such that the map $|X_{\bullet}| \rightarrow \mathcal{M}_g$ is highly connected (see Proposition 5.2 below for the precise meaning).

Before defining X_{\bullet} in Definition 5.1, we recall that the topology of $\mathcal{M}_g(\mathbb{R}^N)$ is defined by the homeomorphism

$$\mathcal{M}_g(\mathbb{R}^N) = \text{Emb}^\partial(W_{g,1}, [0, \infty) \times \mathbb{R}^N) / \text{Diff}^\partial(W_{g,1}),$$

where Emb^∂ denotes the space of embeddings with fixed behaviour near the boundary (in terms of a collar $[0, 1] \times \partial W_{g,1} \rightarrow W_{g,1}$) and Diff^∂ denotes the space of diffeomorphisms which fix a neighbourhood of the boundary pointwise. It is well known (see e.g. [BF81]) that the quotient map from the embedding space is a principal $\text{Diff}^\partial(W_{g,1})$ -bundle.

Definition 5.1. Pick once and for all a coordinate patch $c_0 : \mathbb{R}^{2n-1} \rightarrow S^{2n-1}$. This choice induces for any $W \in \mathcal{M}_g$ a germ of an embedding $[0, 1] \times \mathbb{R}^{2n-1} \rightarrow W$ as in Definition 4.1, and we let $X_p(\mathbb{R}^N)$ be the space of pairs (W, ϕ) where $W \in \mathcal{M}_g(\mathbb{R}^N)$ and $\phi \in \overline{K}_p(W)$, topologised as

$$X_p(\mathbb{R}^N) = (\text{Emb}^\partial(W_{g,1}, [0, \infty) \times \mathbb{R}^N) \times \overline{K}_p(W_{g,1})) / \text{Diff}^\partial(W_{g,1}).$$

This makes $X_\bullet(\mathbb{R}^N)$ into a semisimplicial space augmented over $\mathcal{M}_g(\mathbb{R}^N)$. By the local triviality of the quotient map defining the topology on $\mathcal{M}_g(\mathbb{R}^N)$, the augmentation $X_\bullet(\mathbb{R}^N) \rightarrow \mathcal{M}_g(\mathbb{R}^N)$ is locally trivial with fibres $\overline{K}_\bullet(W_{g,1})$.

Proposition 5.2. *The map $|X_\bullet(\mathbb{R}^N)| \rightarrow \mathcal{M}_g(\mathbb{R}^N)$ induced by the augmentation is $\lfloor (g-3)/2 \rfloor$ -connected.*

Proof. The map is a locally trivial fibre bundle (at least after replacing $\mathcal{M}_g(\mathbb{R}^N)$ with a compactly generated space) with fibre $|\overline{K}_\bullet(W_{g,1})|$ which is $\lfloor (g-5)/2 \rfloor$ -connected, so the claim follows from the long exact sequence in homotopy groups. \square

The direct limit of $X_p(\mathbb{R}^N)$ as $N \rightarrow \infty$ shall be denoted X_p . These form a semisimplicial space augmented over \mathcal{M}_g , and the proposition implies that $|X_\bullet| \rightarrow \mathcal{M}_g$ is also $\lfloor (g-3)/2 \rfloor$ -connected. Next we describe the homotopy type of each space X_p . For example, X_0 is the moduli space of manifolds diffeomorphic to $W_{g,1}$, equipped with an embedding of $H \approx W_{1,1}$. Using cancellation (Corollary 4.5), it is not hard to convince oneself that this is weakly equivalent to \mathcal{M}_{g-1} (for example, by thinking about the functors classified by the two spaces). More generally X_p is weakly equivalent to \mathcal{M}_{g-p-1} , but we need to be precise about the map inducing the homotopy equivalence.

Let $S_1 \subset [0, 1] \times \mathbb{R}^{2n}$ be the manifold obtained from the cylinder $C = [0, 1] \times S^{2n-1}$ by forming the connected sum with an embedded $S^n \times S^n$ along a small disk in C . (For example, we could embed $S^n \times S^n$ as the boundary of a tubular neighbourhood of an embedding $S^n \rightarrow [0, 1] \times \mathbb{R}^{2n}$, although the precise choice does not matter.) This comes with a canonical embedding $W_{1,1} \rightarrow S_1$, and we pick an extension of this to an embedding of $H = W_{1,1} \cup ([0, 1] \times D^{2n-1})$, giving an element $\phi_0 \in \overline{K}_0(S_1)$. (For defining $\overline{K}_\bullet(S_1)$ we use the same coordinate patch $c_0 : \mathbb{R}^{2n-1} \rightarrow \{0\} \times S^{2n-1} \subset \partial S_1$ as in Definition 5.1.) Similarly, the p -fold concatenation $S_p \subset [0, p] \times \mathbb{R}^{2n}$ of S_1 with itself has p canonical embeddings of $W_{1,1}$ which we may extend to p disjoint embeddings of H , giving an element $(\phi_0, \dots, \phi_{p-1}) \in \overline{K}_{p-1}(S_p)$. We shall later be slightly more precise about these choices, but for any such choice we get a map

$$(5.1) \quad \begin{aligned} \mathcal{M}_{g-p} &\longrightarrow X_{p-1} \\ W &\longmapsto (S_p \cup (pe_1 + W), (\phi_0, \dots, \phi_{p-1})), \end{aligned}$$

and the following holds.

Proposition 5.3. *For $g-p \geq 4$, the map (5.1) is a weak equivalence.*

Proof. There is a restriction map from $X_{p-1}(\mathbb{R}^N)$ to the space of embeddings (with fixed behaviour on part of the boundary) of $\{0, \dots, p-1\} \times H$ into $[0, \infty) \times \mathbb{R}^N$. The restriction map is a Serre fibration and in the limit $N = \infty$ the target is contractible, so X_{p-1} is weakly equivalent to any fibre of that fibration, and hence to the subspace of \mathcal{M}_g which consists of manifolds containing the image of the embeddings $\phi_0, \dots, \phi_{p-1}$. If we let $A \subset S_p$ denote the union of these images and a collar neighbourhood of $\{0\} \times S^{2n-1}$, then the inclusion $A \rightarrow S_p$ is an isotopy equivalence. It follows that X_{p-1} is weakly equivalent to the subspace of \mathcal{M}_g consisting of manifolds containing S_p , but by the cancellation theorem (Corollary 4.5), this space is in turn equivalent to \mathcal{M}_{g-p} . \square

Recall that “the” stabilisation map $\mathcal{M}_{g-1} \rightarrow \mathcal{M}_g$ was defined by gluing a submanifold of $[0, 1] \times \mathbb{R}^N$ diffeomorphic to $S^n \times S^n$ with two discs cut out. For large N , the space of such submanifolds is path connected (although not contractible!) so we get a well defined homotopy class. To be precise, we shall use the same submanifold $S_1 \subset [0, 1] \times \mathbb{R}^{2n} \subset [0, 1] \times \mathbb{R}^N$ as above. Together with Proposition 5.3, our next result says that the last face map of X_\bullet is a model for the stabilisation.

Proposition 5.4. *The following diagram is commutative for $p \geq 0$*

$$\begin{array}{ccc} \mathcal{M}_{g-p-1} & \longrightarrow & \mathcal{M}_{g-p} \\ \downarrow & & \downarrow \\ X_p & \xrightarrow{d_p} & X_{p-1}, \end{array}$$

where the vertical maps are given by (5.1) and the top horizontal map is the stabilisation map.

Proof. Starting with $W \in \mathcal{M}_{g-p-1}$ we map it right to $S \cup (e_1 + W) \in \mathcal{M}_{g-p}$ and then down to the element of X_p given by the manifold

$$S_p \cup (pe_1 + (S \cup (e_1 + W))) = S_{p+1} \cup ((p+1)e_1 + W) \subset [0, p+1] \times \mathbb{R}^N$$

equipped with the embeddings $(\phi_0, \dots, \phi_{p-1})$. If instead we map it down to X_p , we get the element with the same underlying manifold but equipped with the embeddings (ϕ_0, \dots, ϕ_p) , and the face map $d_p : X_p \rightarrow X_{p-1}$ then forgets ϕ_p . \square

Finally, we want to show that all face maps $d_i : X_p \rightarrow X_{p-1}$ are homotopic, $i = 0, \dots, p$. For this, we need to be slightly more precise about the choices of $(\phi_0, \dots, \phi_{p-1}) \in \overline{K}_{p-1}(S_p)$ used in the map (5.1). Firstly, the inclusion $S_p \rightarrow S_{p+1}$ induces a map $\overline{K}_\bullet(S_p) \rightarrow \overline{K}_\bullet(S_{p+1})$ which we may assume sends the $\phi_i \in \overline{K}_0(S_p)$ to the elements of $\overline{K}_0(S_{p+1})$ with the same names. Secondly, we may assume that the coordinate patch $c_0 : \mathbb{R}^{2n-1} \rightarrow S^{2n-1}$ extends to an embedding $[0, 1] \times \mathbb{R}^{2n-1} \rightarrow S$ whose image is disjoint from the canonical embedding $W_{1,1} \rightarrow S$. By the argument in the proof of Corollary 4.4, we may then pick a diffeomorphism $\psi : S_2 \rightarrow S_2$ supported in the interior of the complement of the embedded $[0, 2] \times \mathbb{R}^{2n-1}$, which interchanges the two canonical embeddings $W_{1,1} \rightarrow S_2$. We now pick $\phi_1 \in \overline{K}_0(S_2)$ in the same path component as $\psi \circ \phi_0$. (The embedding ϕ_1 will automatically be equal to $\psi \circ \phi_0$ when restricted to $W_{1,1} \subset H$ and we choose the extension to the “tether” $[0, 1] \times D^{2n-1} \subset H$ by isotoping what $\psi \circ \phi_0$ does.) More generally for $p \geq 2$ and $1 \leq i < p$ we let $\psi_{(i-1,i)}$ be the diffeomorphism of $S_p \subset [0, p] \times \mathbb{R}^{2n}$ which acts as ψ inside $[i-1, i+1] \times \mathbb{R}^{2n}$ and is the identity outside. We may then inductively pick $\phi_p \in \overline{K}_0(S_{p+1})$ in the same path component as $\psi_{(p-1,p)} \circ \phi_{p-1}$, and such that ϕ_p is disjoint from the images of ϕ_i and the support of $\psi_{(i-1,i)}$ for $i < p$.

Proposition 5.5. *For $0 \leq p \leq g-5$, all face maps $d_i : X_p \rightarrow X_{p-1}$ are weakly homotopic to one another.*

Proof. Let us focus on the case $p = 1$, the general case being similar. If for $i = 0, 1$ we write f_i for the composition of $d_i : X_1 \rightarrow X_0$ with the weak equivalence $\mathcal{M}_{g-2} \rightarrow X_1$ from Proposition 5.3, we shall construct a homotopy $f_0 \simeq f_1 : \mathcal{M}_{g-2} \rightarrow X_0$. These maps are given by the formula

$$f_i(W) = (S_2 \cup (2e_1 + W), \phi_i).$$

Now the composition of the inclusion $i : S_2 \rightarrow [0, 2] \times \mathbb{R}^N$ with the diffeomorphism $\psi : S_2 \rightarrow S_2$ is an embedding which agrees with i near ∂S_2 . For large N the space of such embeddings is path connected, so we may find an isotopy of embeddings $h_t : S_2 \rightarrow [0, 2] \times \mathbb{R}^N$ from i to $i \circ \psi$, which restricts to the constant isotopy of embeddings of a neighbourhood of ∂S_2 . Then

$$W \mapsto (h_t(S_2) \cup (2e_1 + W), h_t \circ \phi_1).$$

gives a homotopy of maps $\mathcal{M}_{g-2} \rightarrow X_0$ which starts at f_1 and ends at the map $h_1 : W \mapsto (S_2 \cup (2e_1 + W), \psi \circ \phi_1)$, but since $\psi \circ \phi_1 \in \overline{K}_0(W_{2,2})$ is in the same path component as ϕ_0 , the map h_1 is clearly homotopic to f_0 . \square

5.2. The spectral sequence and homological stability. We now prove Theorem 1.2 by induction, using the augmented simplicial space X_\bullet . The induction hypothesis implies that for $p > 0$ the face maps $d_i : X_p \rightarrow X_{p-1}$ induce homology isomorphisms in a range, and the induction step will consist of proving that the augmentation $X_0 \rightarrow X_{-1}$ induces a homology isomorphism in a range which is one larger than the range for $X_2 \rightarrow X_1$. To prove this, we consider the spectral sequence induced by the augmented simplicial space X_\bullet , with E^1 term $E_{p,q}^1 = H_q(X_p)$ for $p \geq -1$ and $q \geq 0$. The differential is given by $d^1 = \sum (-1)^i (d_i)_*$, and the group $E_{p,q}^\infty$ is a subquotient of the relative homology $H_{p+q+1}(X_{-1}, |X_\bullet|)$.

Lemma 5.6. *We have isomorphisms $E_{p,q}^1 \cong H_q(\mathcal{M}_{g-p-1})$ for $-1 \leq p \leq g-5$, with respect to which the differential*

$$H_q(\mathcal{M}_{g-p-1}) \cong E_{p,q}^1 \xrightarrow{d^1} E_{p-1,q}^1 \cong H_q(\mathcal{M}_{g-p})$$

agrees with the stabilisation map for p even, and is zero otherwise. Furthermore, $E_{p,q}^\infty = 0$ for $p+q \leq \lfloor (g-5)/2 \rfloor$.

Proof. Proposition 5.3 identifies $E_{p,q}^1 = H_q(X_p) \cong H_q(\mathcal{M}_{g-p-1})$ and Proposition 5.5 shows that all maps $(d_i)_* : H_q(X_p) \rightarrow H_q(X_{p-1})$ are equal, $i = 0, \dots, p$. Therefore all terms in the differential $d^1 = \sum (-1)^i (d_i)_*$ cancel for p odd, and for p even the term $(d_p)_*$ survives and by Proposition 5.4 is identified with the stabilisation map.

The group $E_{p,q}^\infty$ is a subquotient of the relative homology $H_{p+q+1}(X_{-1}, |X_\bullet|)$, but this vanishes for $p+q+1 \leq \lfloor (g-3)/2 \rfloor$ since the map $|X_\bullet| \rightarrow X_{-1}$ is $\lfloor (g-3)/2 \rfloor$ -connected by Proposition 5.2. \square

Proof of Theorem 1.2. Let us write $a = \lfloor (g-5)/2 \rfloor$. We will use the spectral sequence above to prove that $H_q(\mathcal{M}_{g-1}) \rightarrow H_q(\mathcal{M}_g)$ is an isomorphism for $q \leq a$, assuming we know inductively that for $j > 0$ the stabilisation maps $H_q(\mathcal{M}_{g-2j-1}) \rightarrow H_q(\mathcal{M}_{g-2j})$ are isomorphisms for $q \leq a-j$. By Lemma 5.6, this implies that the differential $d^1 : E_{2j,q}^1 \rightarrow E_{2j-1,q}^1$ is an isomorphism for $0 < j \leq a-q$, and hence that $E_{p,q}^2 = 0$ for $0 < p \leq 2(a-q)$. In particular, the $E_{p,q}^2$ term vanishes in the region given by $p \geq 1$, $q \leq a-1$ and $p+q \leq a+1$, and thus for $r \geq 2$ and $q \leq a$ it follows that differentials into $E_{-1,q}^r$ and $E_{0,q}^r$ vanish. We deduce that for $q \leq a$ we have

$$\begin{aligned} E_{0,q}^\infty &= E_{0,q}^2 = \text{Ker}(H_q(\mathcal{M}_{g-1}) \rightarrow H_q(\mathcal{M}_g)) \\ E_{-1,q}^\infty &= E_{-1,q}^2 = \text{Coker}(H_q(\mathcal{M}_{g-1}) \rightarrow H_q(\mathcal{M}_g)), \end{aligned}$$

and since the group $E_{p,q}^\infty$ vanishes for $p+q \leq a$ we see that the stabilisation map $H_q(\mathcal{M}_{g-1}) \rightarrow H_q(\mathcal{M}_g)$ has vanishing kernel and cokernel for $q \leq a$, establishing the induction step. The statement is vacuous for $g = 1$ and $g = 2$, which starts the induction. \square

Remark 5.7. A similar argument shows that $H_q(\mathcal{M}_{g-1}) \rightarrow H_q(\mathcal{M}_g)$ is surjective for $q = \lfloor (g-3)/2 \rfloor$, but this requires the stronger version of Corollary 4.5 from [Kre99]. Thus the map in Theorem 1.2 is surjective for $g \geq 2k+2$.

Remark 5.8. In the cases $n = 3$ and $n = 7$, the quadratic module $(\pi_n(W_{g,1}), \lambda, \alpha)$ is just \mathbb{Z}^{2g} with its standard symplectic form, as the quadratic form α takes values in the trivial group. In this case, $|K^a(\pi_n(W_{g,1}), \lambda, \alpha)|$ is $\lfloor (g-3)/2 \rfloor$ -connected by [vdKL11, Theorem 4.1]. Using this improvement of Theorem 3.2, the same argument shows that the map in Theorem 1.2 is an isomorphism for $g \geq 2k+2$ (and surjective for $g \geq 2k$) when n is 3 or 7.

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